

**Abstract:**

In this paper, we present several inequalities among various elements of a triangle.

**Key Words:** Nesbitt's Inequality & Cauchy – Inequality

**Introduction:**

Many of the inequalities we have studied and the techniques we have learnt have their direct implications in a class of inequalities known as geometric inequalities. These inequalities explore relations among various geometric elements. When we consider a triangle, we can associate many things with it: angles, sides, area, medians altitudes, circum-radius, in-radius, ex-radii and so on. The classic example is Euler's inequality:  $R \geq 2r$ . Where  $R$  is the circum-radius and  $r$  is the in-radius.

We use the following standard notations for a triangle  $ABC$ :

- ✓  $a = |BC|$ ,  $b = |CA|$ ,  $c = |AB|$ .
- ✓  $S$  is the semi – perimeter of  $ABC$  :  $S = (a+b+c)/2$
- ✓  $R$  is the circum – radius.

We prove the following theorems:

**Theorem 1:**  $abc > 8(s-a)(s-b)(s-c)$ .

**Proof:** We have  $a^2 - (b-c)^2 \leq a^2$  and equality holds if and only if  $b=c$ .

Similar inequalities hold :  $b^2 - (c-a)^2 \leq b^2$ ,  $c^2 - (a-b)^2 \leq c^2$ .

$$\begin{aligned} \text{Hence, } abc &\geq \sqrt{a^2 - (b-c)^2} \sqrt{b^2 - (c-a)^2} \sqrt{c^2 - (a-b)^2} \\ &= (a+b-c)(b+c-a)(c+a-b) \\ &= 8(s-a)(s-b)(s-c). \end{aligned}$$

Equality holds if and only if  $a=b=c$ .

**Theorem 2:**  $abc < \sum a^2(s-a) \leq 3/2 abc$ .

**Proof:** We have  $2 \sum a^2(s-a) = a^2(b+c-a) + b^2(c+a-b) + c^2(a+b-c)$   
 $= \sum a^2b + \sum ab^2 - \sum a^3$

On the other hand, we also see that

$$(b+c-a)(c+a-b)(a+b-c) = (c^2 - a^2 - b^2 + 2ab)(a+b-c) = a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 - a^3 - b^3 - c^3 - 2abc$$

Thus, we obtain  $2 \sum a^2(s-a) = (b+c-a)(c+a-b)(a+b-c) + 2abc$ .

Since  $a, b, c$  are the sides of a triangle, we know that  $b+c-a > 0$ ,  $c+a-b > 0$  and  $a+b-c > 0$ .

$$\text{Hence, } abc < \sum a^2(s-a).$$

Now using (1), we get  $2 \sum a^2(s-a) \leq abc + 2abc = 3abc$ ,

Which proves the right hand side inequality.

Again, we may use Stolarsky's theorem.

Considering  $P(x, y, z) = \sum_{\text{cyclic}} x^2(y+z-x) - 2xyz$ ,

We see that it is a homogeneous polynomial of degree 3 and

$P(1, 1, 1) = 1$ ,  $P(1, 1, 0) = 0$ ,  $P(2, 1, 1) = 0$ .

Hence,  $P(a, b, c) > 0$ , giving the left – side inequality. On the other hand, the polynomial

$$Q(x, y, z) = 3xyz - \sum_{\text{cyclic}} x^2(y+z-x),$$

Gives  $Q(1, 1, 1) = 0$ ,  $Q(1, 1, 0) = 0$  and  $Q(2, 1, 1) = 2$ .

Thus  $Q(a, b, c) > 0$ , and we get the right – side inequality.

**Theorem 3:**  $\frac{3}{2} \leq \sum \frac{a}{b+c} < 2$ . Equality holds on the left if and only if  $a=b=c$ .

**Proof:** The first part of the above inequality is equivalent to

$$\begin{aligned} \frac{9}{2} &\leq \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b} + 1 \\ &= (a+b+c) \left( \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \right) \end{aligned}$$

If we introduce  $a+b=x$ ,  $b+c=y$  and  $c+a=z$ , this reduces to  $9 \leq (x+y+z) \left( \frac{1}{y} + \frac{1}{z} + \frac{1}{x} \right)$

Which is a consequence of the AM – GM inequality. Suppose  $c$  is the largest among  $a, b, c$ .

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By the symmetry, we may assume  $a \leq b \leq c$ . In this case

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\leq \frac{a}{a+c} + \frac{c}{c+a} + \frac{c}{a+b} \\ &= 1 + \frac{c}{a+b} \\ \sum \frac{a}{b+c} &< 2. \end{aligned}$$

Since  $c < a + b$  by the triangle inequality.

**Proof:** The left hand side of the above inequality is generally known as Nesbitt's inequality. There are a variety of ways of proving this. We give two such proofs.

(i) Using the Cauchy – Schwarz inequality, we have

$$\begin{aligned} (a+b+c)^2 &= (\sum a)^2 \\ &= \left( \sum \sqrt{\frac{a}{b+c}} \sqrt{a(b+c)} \right)^2 \\ (a+b+c)^2 &\leq \left( \sum \frac{a}{b+c} \right) \sum a(b+c) \end{aligned}$$

This gives

$$\begin{aligned} \sum \frac{a}{b+c} &\geq \frac{(a+b+c)^2}{\sum a(b+c)} \\ &= \frac{(a+b+c)^2}{2(ab+bc+ca)} \\ \sum \frac{a}{b+c} &\geq \frac{3(ab+bc+ca)}{2(ab+bc+ca)} = \frac{3}{2} \end{aligned}$$

since  $(a+b+c)^2 \geq 3(ab+bc+ca)$ .

(ii) we may assume  $a \leq b \leq c$ , since the inequality is symmetric in  $a, b, c$ .

This implies that

$$\frac{1}{b+c} \leq \frac{1}{c+a} \leq \frac{1}{a+b}$$

Using rearrangement inequality, we obtain

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\geq \frac{a}{c+a} + \frac{b}{a+b} + \frac{c}{b+c}, \\ \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &\geq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \end{aligned}$$

Adding these two, we obtain

$$2 \left( \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right) \geq 3.$$

This gives the desired inequality.

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